Realizations of non-Abelian statistics (fractionalized degrees of freedom)

Xiao-Gang Wen (2018/5 KITS)

Topological order = pattern of many-body entanglement $\rightarrow$ fractionalization and protected qubits Wen PRB **40** 7387 (89)

- Topological orders are new states of quantum matter beyond symmetry breaking.
- They have fractional quantum numbers and fractional statistics (including **non-Abelian statistics**).
- Non-Abelian statistics has fractional degrees of freedom.
  Quantum dimension d:

spin-1/2: $d = 2$ degrees of freedom
Ising anyon: $d = \sqrt{2}$ degrees of freedom
Fibonacci anyon: $d = (\sqrt{5} + 1)/2$ degrees of freedom

- Let $D_{\text{trap}}(N)$ be the number of degenerate states of N trapped quasi-particles. The quantum dimension of the quasi-particle is

$$d = \lim_{N \to \infty} [D_{\text{trap}}(N)]^{1/N}$$

- The degeneracy of trapped non-Abelian anyons is robust against all perturbations $\rightarrow$ Fault tolerant topo. quantum computation.

First examples of non-Abelian topological order

Let $\chi_m(\{z_i\})$ be the many-body wavefunction of m filled Landau levels, where $z_i = x_i + \mathrm{i}\, y_i$.

- Laughlin $\nu = 1/3$ state (Abelian): $\Psi_{\nu=1/3} = (\chi_1)^3$
- $SU(m)_k$ state via slave-particle — Wen PRL **66** 802 (1991)

$$\Psi_{SU(3)_2} = (\chi_2)^3,\ \nu = \frac{2}{3}; \quad \Psi_{SU(2)_2} = \chi_1(\chi_2)^2,\ \nu = \frac{1}{2};$$

$\to SU(2)_2$, $SU(3)_2$ Chern-Simons theory $\to$ non-abelian statistics

- Pfaffien state via CFT correlation — Moore-Read NPB **360** 362 (1991)

$$\Psi_{\text{Pfa}} = \mathcal{A}[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots]\prod(z_i - z_j)^2 \mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \quad \nu = \frac{1}{2}$$

- The $\Psi_{SU(2)_2}$ and Ψ_{Pfa} states have **Ising non-abelian anyons**
- The $\Psi_{SU(3)_2}$ state has **Fibonacci non-abelian anyon.**

- Fibonacci non-abelian anyons can do universal quantum computation, while Ising non-abelian anyons cannot.

How to realize non-Abelian FQH states

- $\Psi_{\nu=2/5} = (\chi_1)^2\chi_2$ can be realized if the first 2 LLs are degenerate
  $\Psi_{SU(2)_2} = \chi_1(\chi_2)^2$ can be realized if the first 3 LLs are degenerate
  $\Psi_{SU(3)_2} = (\chi_2)^3$ can be realized if the first 4 LLs are degenerate

- $\Psi_{SU(2)_2} = \chi_1(\chi_2)^2$ state contains a neutral fermionic quasiparticle ψ. By condensing ψ into different IQH states, we can obtain Ψ_{Pfa}, $\Psi_{\text{PH-Pfa}}$, $\Psi_{\overline{\text{Pfa}}}$, $\Psi_{\overline{SU(2)_2}}$, etc.

- $\Psi_{\text{PH-Pfa}}$ is realized in the 2nd LL.

- **Realizing non-Abelian FQH state in bi-layer systems**

- Starting with (nnm) state

$$\Phi_{nnm} = \prod (z_i - z_j)^n (w_i - w_j)^n (z_i - w_i)^m e^{-\frac{1}{4}\sum |z_i|^2 + |w_i|^2}$$

where $n = odd$ for fermionic electron.

- Increase interlayer tunneling to induce an one-layer FQH state

States induced by interlayer tunneling: $\mathcal{A}(331)$ and $\mathcal{A}(330)$

- The tunneling-induced one-layer state via anti-symmetrization:
$$\Psi_{\mathcal{A}(nnm)}(x_i) = \mathcal{A}[\prod (z_i - z_j)^n (w_i - w_j)^n (z_i - w_j)^m].$$
- **Characterize them with pattern-of-zeros**: Wen-Wang arXiv:0801.3291
(similar to s-wave, p-wave, etc of superconducting states)

	S_2	S_3	S_4	S_5	$\cdots$
$\Psi_{\mathcal{A}(331)}$	1	5	10	18	$\cdots$
$\Psi_{\mathcal{A}(330)}$	1	3	6	12	$\cdots$
$\prod(z_i - z_j)^n$	n	$3n$	$6n$	$10n$	$\cdots$

S_a = total relative angular momentum of a electrons.

- **Obtain their properties using POZ** → Spectrum of gapless edge excitations. The ground state has a total angular momentum M_0. The chiral edge excitations have higher angluar mementa $M_0 + m$. $D_{\text{edge}}(m)$ = number of edge excitations at $M_0 + m$.
- How to compute $D_{\text{edge}}(m)$?
$D_{\text{edge}}(m)$ = number of anti-symmetric holomorphic functions $\Psi(z_i)$ whose n-electron relative angular momentum $\tilde{S}_n \geq S_n$.

The edge spectrum $D_{\text{edge}}(m)$

m :	0	1	2	3	4	$\cdots$	c	remark
$\Psi_{\mathcal{A}(331)}$	1	1	3	5	10	$\cdots$	$\frac{3}{2}$	Edge: chiral Majorana fermion. Bulk: Ising anyon
$\Psi_{\mathcal{A}(330)}$	1	1	3	6	13	$\cdots$	2	
$\prod(z_i - z_j)^n$	1	1	2	3	5	P_m	1	Edge chiral complex fermion

- The edge spectrum $D_{\text{edge}}(m)$ is described by **central charge** c.

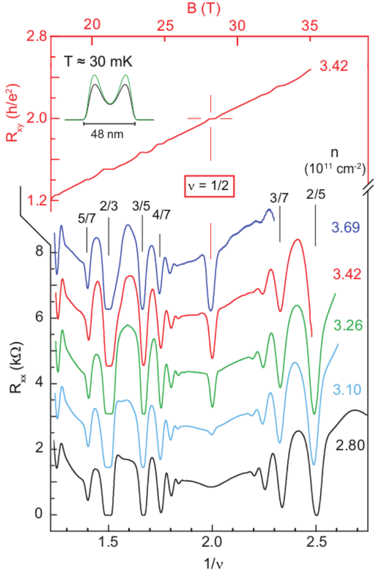
  For $\prod(z_i - z_j)^m$: $P_m \sim \frac{1}{4m\sqrt{3}} e^{\pi\sqrt{\frac{2m}{3}}} \sim e^{c\pi\sqrt{\frac{2m}{3}}}$ with $c = 1$.
- The central charge can be measured by specific heat $C = c\frac{\pi}{6}\frac{k_B^2 T}{v\hbar}$ or quantized thermal Hall conductivity $\kappa_{xy} = c\frac{\pi}{6}\frac{k_B^2 T}{\hbar}$
- The edge spectrum $D_{\text{edge}}(m)$ = finger print for FQH states:

- $D_{\text{edge}}(m)$ = partition number $\to \Psi_{\nu=1/m}$ is an Abelian state.
- $\Psi_{\mathcal{A}(331)} = \Psi_{\text{Pfa}}$ has Ising non-Abelian anyon (Z_2 parafermion non-Abelian statistics) $\leftrightarrow$ Edge chiral Majorana fermion
- $\Psi_{\mathcal{A}(330)}$ has Z_4 parafermion non-Abelian statistics. Blok-Wen Nucl. Phys. B374, 615 (92); Read-Rezayi cond-mat/9809384

Bilayer FQH in a wide quantum well (width = 48nm)

- For very large interlayer tunneling, we get a single-layer compressible state at $\nu = 1/2$.
- For very small interlayer tunneling, we get a bi-layer (331) state.
- In between, we may get the Z_2 parafermion non-Abelian state.

- To get (331) state from $\nu = 1/2$ FL state, we need a d-wave pairing → impossible.
- p-wave pairing on $\nu = 1/2$ FL state gives us Z_2 parafermion non-Abelian state.
- *With less interlayer tunneling, can we see Z_2 parafermion → (331) transition?*



Shabani, Shayegan, etal arXiv:1306.5290

Two-component states in bi-layer systems

Consider two-component bi-layer FQH states, such as
$\Psi(z_i, w_i) = \prod(z_i - z_j)^n (w_i - w_j)^n (z_i - w_j)^m$

- The pattern-of-zeros description of two-component states:
  S_{ab} = the total relative angular momentum for a cluster of a electron in layer-1 and b electron in layer-2.
- For the (nnm) state $S_{ab} = n\frac{a(a-1)}{2} + n\frac{b(b-1)}{2} + mab$:
- There are other more interesting FQH states described by different POZs, such as $\nu = \frac{4}{5}, \frac{4}{7}$ bi-layer states: Barkeshli-Wen arXiv:0906.0341

$\Psi^{\nu=1/2}_{(331)}$, $c = 2$

S_{ab}	0	1	2	3
0	0	0	3	9
1	0	1	5	12
2	3	5	10	18
3	9	12	18	27

$\Psi^{\nu=4/5}_{SU(3)_2/U^2(1)}$, $c = 3\frac{1}{5}$

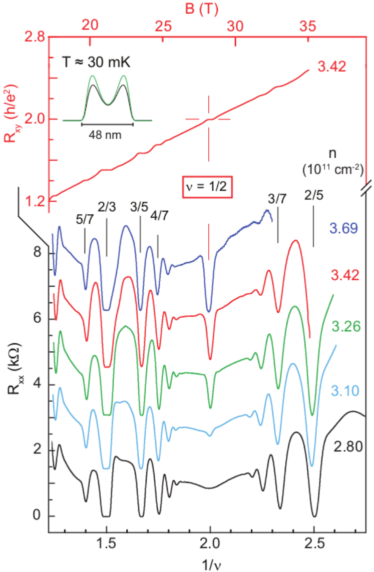
S_{ab}	0	1	2	3
0	0	0	1	5
1	0	1	2	7
2	1	2	4	9
3	5	7	9	15

Compare to the $\nu = \frac{2}{5} + \frac{2}{5}$ state, the $\nu = 4/5$ state has the same intralayer avoidance and a stronger interlayer avoidance.

Fibonacci non-Abelian statistics in wide quantum wells ?

- Just like $(\chi_2)^3$ state, the $\Psi_{SU(3)_2/U^2(1)}$ state also have Fibonacci non-Abelian anyon
- $\nu = 4/5$ FQH state was observed in bi-layer systems (wide quantum wells).
  Is it a Fibonacci FQH state that can do universal topological quantum computation?

Shabani, Shayegan, etal arXiv:1306.5290



Half-qubit on extrinsic defect

- Extrinsic defects: vortex, end of line, dislocation, etc
- Extrinsic defects can carry half-qubit $d = \sqrt{2}$

\- An extrinsic defect carrying half-qubit is not an non-Abelian anyon, since they cannot move. Such extrinsic defect can appear in 3D.

Teo-Kane, "Majorana Fermions and Non-Abelian Statistics in Three Dimensions", 09

\- A point-like excitation carrying half-qubit is an Ising non-Abelian anyon. Such non-Abelian anyon can only appear in 2D.

Wen PRL **66** 802 (1991); Moore-Read NPB **360** 362 (1991)

If 2D bulk has Ising non-Abelian anyon, then the edge must has 1D chiral **Majorana fermion**. Wen PRL **70** 355 (93)

- For some free fermion systems, certain extrinsic defect can carry half-qubit → **Majorana zero mode** (not Majorana fermion).

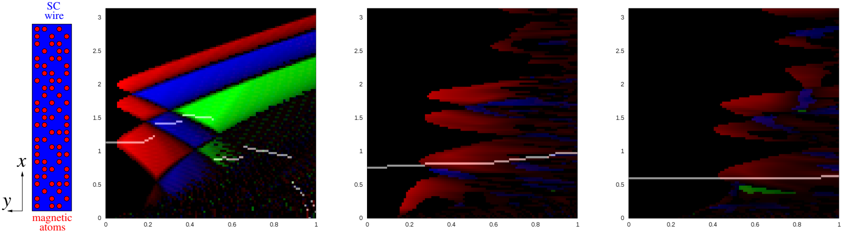
\- If 2D bulk has particle carrying *Majorana zero mode* , then the edge must has 1D chiral *Majorana fermion*.

Realizing Majorana zero mode (MZM)

- Vortex in 2D $p + \mathrm{i}p$ superconductor. Read-Green, cond-mat/9906453
- End of p-wave superconducting chain. Kiteav, cond-mat/0010440
- Vortex in 2D SC-TI junction. Fu-Kane, arXiv:0707.1692

- FeSe wire of width $5 \sim 10$nm, springkled with magnetic atoms. The MEC can induce Yu-Shiba states, forming a TSC with MZM

L. Yu, Acta Phys. Sin. **21** 75 (65) Ji-Wen arXiv:1804.10198



Uniform: $J_c = \Delta_{SC}$, $\Delta_{TSC} = 0.9\Delta_{SC}$

Random: $\Delta\mu = 0.5t$, $J_c = 0.25t$, $\Delta_{TSC} = 0.4\Delta_{SC}$

$\Delta\mu = t$, $J_c = 0.4t$, $\Delta_{TSC} = 0.3\Delta_{SC}$